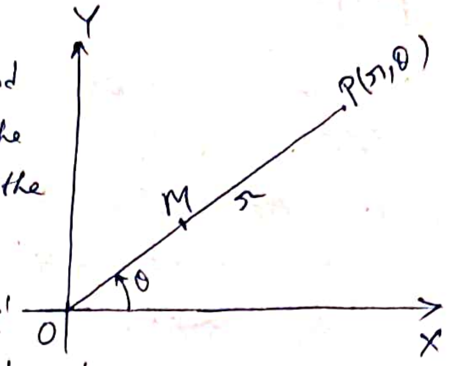
Two dimensional motion, Velocity and acceleration in Polar co-ordinates :

(4.1) (H'77,'83,'94) Find the radial and cross radial (or transversed components of velocity and accl^n in terms of polar co-ordinates of a particle moving in a plane..



A set of rectangular axes XOX' and YOY' are taken in the plane of motion of the particle. Let $P(r, \theta)$ be the position of the particle at time t.

Let $\hat{i}, \hat{j}$ be unit vectors along OX & OY respectively. $\hat{\alpha}, \hat{\beta}$ be unit vectors along OP and $\perp^r$ to OP (in the sense of θ increasing).

OM = 1, is taken on OP. $\therefore \vec{OM} = \hat{\alpha}$ $\therefore$ Co-ordinates of M are

$(1.\cos\theta, 1.\sin\theta) = (\cos\theta, \sin\theta)$, $\therefore \vec{OM} = \hat{i}\cos\theta + \hat{j}\sin\theta$

$$\therefore \hat{\alpha} = \hat{i}.\cos\theta + \hat{j}\sin\theta .$$

Similarly $\hat{\beta} = \hat{i}\cos(\pi/2 + \theta) + \hat{j}\sin(\pi/2 + \theta) = -\hat{i}\sin\theta + \hat{j}\cos\theta$

$$\frac{d\hat{\alpha}}{d\theta} = -\hat{i}\sin\theta + \hat{j}\cos\theta = \hat{\beta} . \quad \therefore \frac{d\hat{\alpha}}{dt} = \frac{d\hat{\alpha}}{d\theta}.\dot{\theta} = \hat{\beta}\dot{\theta}.$$

$$\frac{d\hat{\beta}}{d\theta} = -i\cos\theta - j\sin\theta = -\hat{\alpha}, \quad \therefore \frac{d\hat{\beta}}{dt} = -\hat{\alpha}\dot{\theta}$$

Let V_r and V_θ be the components of vel. along OP and $\perp^r$ to OP.

$$\therefore \vec{V} = (V_r)\hat{\alpha} + (V_\theta)\hat{\beta} = \frac{d(\vec{OP})}{dt} = \frac{d}{dt}(r\hat{\alpha}) = \frac{dr}{dt}\hat{\alpha} + r\frac{d\hat{\alpha}}{dt},$$

$$= \frac{dr}{dt}.\hat{\alpha} + r\frac{d\hat{\alpha}}{d\theta}.\frac{d\theta}{dt} = \frac{dr}{dt}\hat{\alpha} + r\frac{d\theta}{dt}\hat{\beta}.$$

$$\boxed{\begin{aligned} \therefore \quad V_r &= \frac{dr}{dt} \text{ or } \dot{r} \\ V_\theta &= r\frac{d\theta}{dt} \text{ or } r\dot{\theta} \end{aligned}}$$

Let f_r and f_θ be radial and cross radial comps of accl^n

$$\therefore \vec{f} = f_r\hat{\alpha} + f_\theta\hat{\beta} = \frac{d}{dt}(\vec{V}) = \frac{d}{dt}(\dot{r}\hat{\alpha} + r\dot{\theta}\hat{\beta})$$

$$= \ddot{r}\hat{\alpha} + \dot{r}\dot{\hat{\alpha}} + \dot{r}\dot{\theta}\hat{\beta} + r\ddot{\theta}\hat{\beta} + r\dot{\theta}\dot{\hat{\beta}}$$

$$= \ddot{r}\hat{\alpha} + \dot{r}\hat{\beta}\dot{\theta} + \dot{r}\dot{\theta}\hat{\beta} + r\ddot{\theta}\hat{\beta} - r\dot{\theta}\hat{\alpha}\dot{\theta}$$

$$= (\ddot{r} - r\dot{\theta}^2)\hat{\alpha} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\beta}$$

$$\therefore f_r = \ddot{r} - r\dot{\theta}^2, \quad f_\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta} = \frac{1}{r}(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}) = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$$

$$\boxed{\begin{aligned} \therefore \quad f_r &= \ddot{r} - r\dot{\theta}^2 \\ f_\theta &= \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) \end{aligned}}$$

Ex-1

A particle describes the curve $r = ae^{\theta}$ in such a manner that its accel^n has no radial comp. Show that its angular vel. is const and that the magnitudes of vel. and accel^n at a point are each proportional to radius vector r.

Ans:- Radial accel^n at (r,θ) at any time $t = 0$. (given)

$$\therefore \ddot{r} - r(\dot{\theta})^2 = 0 \quad \text{---(1)}$$

$$r = ae^{\theta}, \quad \dot{r} = ae^{\theta}.\dot{\theta} = r\dot{\theta}$$

$$\ddot{r} = \dot{r}\dot{\theta} + r\ddot{\theta} = r\dot{\theta}^2 + r\ddot{\theta}$$

from (1) $\quad r\dot{\theta}^2 + r\ddot{\theta} - r\dot{\theta}^2 = 0$ or, $r\ddot{\theta} = 0$ or, $\ddot{\theta} = 0$

$\therefore \dot{\theta} = \text{const} = \omega$ (say) (proved)

Radial velocity $v_r = \dot{r} = r\dot{\theta} = r\omega$

Cross-radial vel. $v_\theta = r\dot{\theta} = r\omega$

$\therefore$ Mag. of vel. $= \sqrt{v_r^2 + v_\theta^2} = \sqrt{r^2\omega^2 + r^2\omega^2} = \sqrt{2}\, r\omega$

$\therefore$ magnitude of velocity is $\propto r$ (proved)

Cross radial accel^n $f_\theta = \frac{1}{r}\frac{d(r^2\dot{\theta})}{dt} = \frac{1}{r}\frac{d(r^2\omega)}{dt} = \frac{\omega}{r}.2r\dot{r}$

$$= 2\omega^2 r \quad [\because \dot{r} = r\omega]$$

We have $f_r = 0$. (given)

Magnitude of accel^n $= \sqrt{f_r^2 + f_\theta^2} = \sqrt{4\omega^4 r^2} = 2\omega^2 r$, which is proportional to r.

Ex-2 If the path of a particle is the curve $r = ae^{\theta\cot\alpha}$ and if the radius vector of the particle has a const angular vel, show that the resultant accel^n of the particle makes an angle 2α with the radius vector and is of magnitude $\frac{v^2}{r}$, v is the speed of the particle.

Ans Let $P(r,\theta)$ be the position of the particle at time t.

Angular velocity about the pole $= \dot{\theta} = \text{const} = \omega$ (say)

$$r = ae^{\theta\cot\alpha}, \quad \therefore \dot{r} = a(\cot\alpha)e^{\theta\cot\alpha}.\dot{\theta} = (\cot\alpha)\, r\omega$$

$$\ddot{r} = (\cot\alpha)\omega.\dot{r} = (\cot^2\alpha)\,\omega^2 r$$

Radial velocity $v_r = \dot{r} = \cot\alpha . r\omega$

Cross-radial velocity $v_\theta = r\dot{\theta} = r\omega$

$\therefore v^2 = v_r^2 + v_\theta^2 = r^2\omega^2\cot^2\alpha + r^2\omega^2 = r^2\omega^2(1+\cot^2\alpha) = r^2\omega^2\operatorname{cosec}^2\alpha$

Radial accel^n $f_r = \ddot{r} - r\dot{\theta}^2 = (\cot^2\alpha)\omega^2 r - r\omega^2 = \omega^2 r(\cot^2\alpha - 1)$

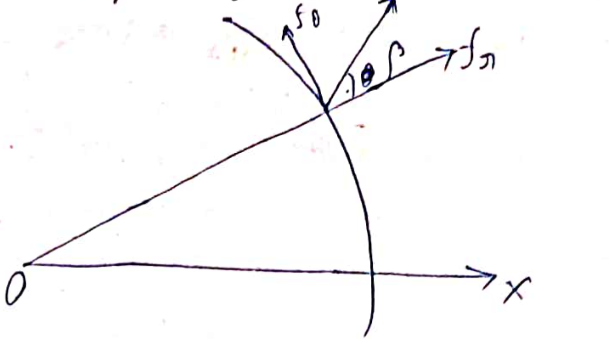
Cross-radial accel^n $f_\theta = \frac{1}{r}\frac{d(r^2\dot{\theta})}{dt} = \frac{1}{r}\frac{d(r^2\omega)}{dt} = \frac{\omega}{r}.2r\dot{r} = 2\omega^2\cot\alpha . r$

Resultant accel^n $= \sqrt{f_r^2 + f_\theta^2} = \sqrt{\omega^4 r^2(\cot^2\alpha - 1)^2 + 4\omega^4 r^2 \cot^2\alpha}$

$= \omega^2 r\sqrt{(\cot^2\alpha - 1)^2 + 4\cot^2\alpha} = \omega^2 r\sqrt{(\cot^2\alpha+1)^2} = \omega^2 r(\cot^2\alpha + 1)$

$= \omega^2 r \operatorname{cosec}^2\alpha = \dfrac{\omega^2 r^2 \operatorname{cosec}^2\alpha}{r} = \dfrac{v^2}{r}.$

Let the resultant acceleration makes an angle β with the radius vector.



$\therefore \tan\beta = \dfrac{f_\theta}{f_r} = \dfrac{2\omega^2 r\cot\alpha}{r\omega^2(\cot^2\alpha - 1)}$

$= \dfrac{2\cot\alpha}{\cot^2\alpha - 1} = \dfrac{2\tan\alpha}{1-\tan^2\alpha} = \tan 2\alpha$

$\therefore \beta = 2\alpha$ (proved)

Ex-3 A ~~point~~ particle moves on a plane with constant linear velocity ωa and its angular velocity about the pole is $\frac{\omega r}{a}$. Show that its accel^n is equal to $3\omega^2 r$.

Let $P(r,\theta)$ be the position of the particle at time t.

Radial vel. $v_r = \dot r$, Cross radial vel. $= v_\theta = r\dot\theta = r.\dfrac{r\omega}{a} = \dfrac{\omega r^2}{a}$

Linear Velocity $= \sqrt{v_r^2 + v_\theta^2} = \sqrt{\dot r^2 + \dfrac{\omega^2 r^4}{a^2}} = \omega a$ (given)

$\therefore \dot r^2 + \dfrac{\omega^2 r^4}{a^2} = \omega^2 a^2$. or, $\dot r^2 = \omega^2 a^2 - \dfrac{\omega^2 r^4}{a^2} = \omega^2\left(\dfrac{a^4 - r^4}{a^2}\right)$

$\therefore \dot r = \pm \dfrac{\omega\sqrt{a^4 - r^4}}{a}$

$\ddot r = \pm \dfrac{\omega}{a}\cdot\dfrac{1}{2}\left(\sqrt{a^4-r^4}\right)^{-1}(-4r^3\dot r) = \mp \dfrac{2r^3\omega}{a}\,\dot r\,\dfrac{1}{\sqrt{a^4-r^4}}$

$= \mp \dfrac{2r^3\omega}{a\sqrt{a^4-r^4}}\cdot\left(\pm\dfrac{\omega\sqrt{a^4-r^4}}{a}\right) = -\dfrac{2\omega^2}{a^2}r^3$

Radial accel^n $f_r = \ddot r - r\dot\theta^2 = -\dfrac{2\omega^2 r^3}{a^2} - r.\dfrac{\omega^2 r^2}{a^2} = -\dfrac{3\omega^2 r^3}{a^2}$

Cross radial accel^n $f_\theta = \dfrac{1}{r}\dfrac{d}{dt}(r^2\dot\theta) = \dfrac{1}{r}\dfrac{d}{dt}\left(r^2.\dfrac{\omega r}{a}\right) = \dfrac{\omega}{a}\cdot\dfrac{1}{r}\cdot 3r^2\dot r$

$= \dfrac{3\omega r}{a}\left(\pm\dfrac{\omega\sqrt{a^4-r^4}}{a}\right) = \pm\dfrac{3\omega^2 r\sqrt{a^4-r^4}}{a^2}$

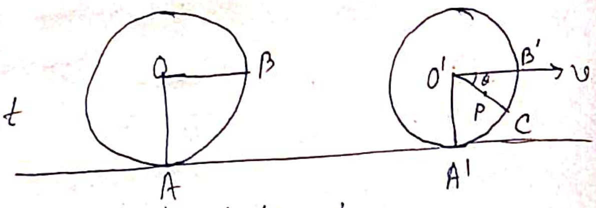
Magnitude of accel^n $= \sqrt{f_r^2 + f_\theta^2} = \sqrt{\dfrac{9\omega^4 r^6}{a^4} + \dfrac{9\omega^4 r^2(a^4-r^4)}{a^4}}$

$= \dfrac{3\omega^2 r}{a^2}\sqrt{r^4 + a^4 - r^4}$

$= \dfrac{3\omega^2 r}{a^2}.a^2 = 3\omega^2 r$ (proved)

Ex-4 An insect crawls at a const. rate u along the spoke of a

Ch. 93 cart wheel of radius a. The cart moving with a const vel. v by pure rolling. Find the accel^n of the insect along and perp. to the spoke.



Let O be the centre of the wheel and A be the pt of contact initially. Let O' be the position of the centre and A' be the pt of contact at time t.

Let P be the position of the insect on the spoke O'C making an angle θ with the horizontal direction O'B'. Let O'P = r,

$\angle B'O'C = \theta$.

$\because$ the insect crawls along the spoke with const vel. u,

$\therefore \frac{dr}{dt} = u$.

$\because$ the wheel rolls, the vel. of the pt. of contact = 0.

$\therefore v - a\dot\theta = 0 \qquad \therefore \dot\theta = \frac{v}{a}$.

Accel^n of the insect along the spoke $= \ddot r - r\dot\theta^2 = 0 - r\frac{v^2}{a^2} \quad \left[\begin{array}{l}\because \dot r = u\\ \therefore \ddot r = 0\end{array}\right]$

$= -\frac{v^2 r}{a^2}$.

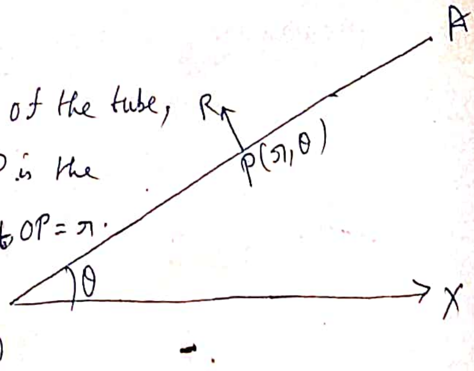
Accel^n of the insect $\perp^r$ to the spoke $= \frac{1}{r}\frac{d}{dt}(r^2\dot\theta) = \frac{1}{r}\cdot\frac{d}{dt}\left(r^2\cdot\frac{v}{a}\right)$

$= \frac{v}{a}\cdot\frac{1}{r}\cdot 2r\dot r = \frac{2v}{a}\cdot u$.

Ex-5 A st. smooth tube revolves with angular vel. ω in a hori-

92 zontal plane about one extremity which is fixed. If at zero time the particle starts with no initial vel. from a pt inside the tube at distance a from the fixed end, find the distance of the particle and the normal pressure of the tube at time t.

If the length of the tube be b, show that the direction in which the particle flies out is inclined to the tube at an angle $\tan^{-1}\frac{b}{\sqrt{b^2-a^2}}$.



Let OX be the initial ~~line~~ position of the tube, and OA be the position ~~po~~ at time t. P is the position of the particle at that time, OP = r.

$\angle POX = \theta$, R = ~~horizontal~~ normal pressure of the tube.

m = mass of the particle.

The equations of motion along and $\perp^r$ to OP are,

$$m(\ddot{r} - r\dot{\theta}^2) = 0 \quad \text{---}(1)$$

$$m\left\{\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\right\} = R \quad \text{--}(2)$$

from (1) $\ddot{r} - r\dot{\theta}^2 = 0$, or, $\ddot{r} - r\omega^2 = 0$.

Let $r = e^{\lambda t}$ be a sol^n of the equation.

$\therefore$ auxiliary equation is $\lambda^2 - \omega^2 = 0 \quad \therefore \lambda = \pm\omega$

$\therefore$ The general sol^n is $r = c_1 \cosh \omega t + c_2 \sinh \omega t$

$\therefore \dot{r} = c_1\omega \sinh \omega t + c_2 \omega \cosh \omega t$

When $t=0$, $r = a$, $\therefore a = c_1 \cdot 1 + c_2 \cdot 0 \quad \therefore c_1 = a$

When $t=0$, $\dot{r} = 0 \quad \therefore 0 = 0 + c_2\omega \cdot 1 \quad \therefore c_2 = 0$.

$\therefore r = a \cosh \omega t$.

This gives the ~~normal pressure~~ distance of the particle at time t. We have, $\dot{r} = a\omega \sinh \omega t$

from (2) $R = \frac{m}{r} \cdot \frac{d(r^2\omega)}{dt} = \frac{m}{r} \cdot \omega \cdot 2r\dot{r}$

$= 2m\omega\, a\omega \sinh \omega t = 2m\omega^2 a \sinh \omega t$

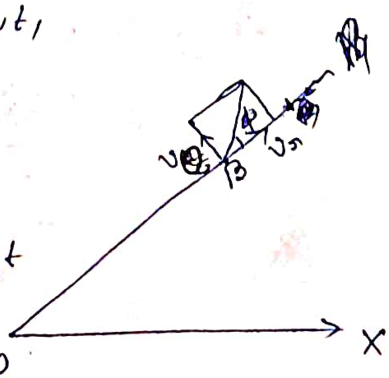
This gives the normal pressure at time t.

Let the particle reaches the end of the tube at time t_1, then $r = b$.

$\therefore b = a\cosh \omega t_1$, $\therefore \cosh \omega t_1 = \frac{b}{a}$, $\therefore \sinh \omega t_1 = \sqrt{\frac{b^2}{a^2} - 1}$
$= \frac{1}{a}\sqrt{b^2 - a^2}$

At $t = t_1$, $V_r = [\dot{r}]_{t=t_1} = a\omega \sinh \omega t_1$

$V_\theta = [r\dot{\theta}]_{t=t_1} = b\omega$



At the end the particle flies out in the direction of the resultant vel. Let the resultant vel. makes an angle ϕ with the tube.

$\therefore \tan\phi = \frac{V_\theta}{V_r} = \frac{b\omega}{a\omega \sinh \omega t_1} = \frac{b}{a \cdot \frac{1}{a}\sqrt{b^2 - a^2}} = \frac{b}{\sqrt{b^2 - a^2}}$

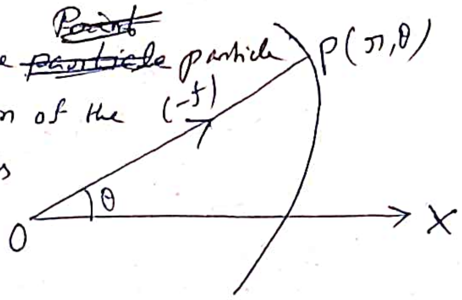
$\therefore \phi = \tan^{-1}\frac{b}{\sqrt{b^2 - a^2}}$.

Ex-6 A ~~point~~ particle starts from the origin in the direction of the initial

83, line with vel. $\frac{f}{\omega}$ and moves with constant angular vel. ω about the origin and with constant negative radial accel^n $(-f)$, Prove that the equ. of the path is $\omega^2 r = f(1-e^{-\theta})$.

Also show that the rate of growth of radial vel. is never +ve and tends to zero.

Let $P(r,\theta)$ be the position of the ~~Point~~ ~~particle~~ particle at time t. The equation of motion of the particle in the radial direction is



$$\ddot{r} - r\dot{\theta}^2 = -f$$

or, $\ddot{r} - r\omega^2 = -f \quad \text{---(1)}$

for C.F. we solve, $\ddot{r} - r\omega^2 = 0 \quad ..(2)$

Let $r = e^{\lambda t}$ be a sol^n of (2)

$\therefore$ The auxiliary equation is $\lambda^2 - \omega^2 = 0$, $\therefore \lambda = \pm\omega$.

$$\therefore C.F = c_1 e^{\omega t} + c_2 e^{-\omega t}.$$

Equation (1) can be written as, $(D^2-\omega^2)r = -f$ $\left[D \equiv \frac{d}{dt}\right]$

$$\therefore P.I = \frac{1}{D^2-\omega^2}(-f) = \frac{f}{\omega^2\left(1-\frac{D^2}{\omega^2}\right)} 1$$

$$= \frac{f}{\omega^2}\left[1 + \frac{D^2}{\omega^2} + \dots\right]1 = \frac{f}{\omega^2}.$$

$\therefore$ G.S. of (1) is $r = c_1 e^{\omega t} + c_2 e^{-\omega t} + \frac{f}{\omega^2}$.

$$\dot{r} = c_1 \omega e^{\omega t} - c_2 \omega e^{-\omega t}$$

at $t=0$, $r=0$.

$$\therefore 0 = c_1 + c_2 + \frac{f}{\omega^2} \quad \text{----(3)}$$

at $t=0$, $\dot{r} = \frac{f}{\omega}$

$$\therefore \frac{f}{\omega} = c_1\omega - c_2\omega$$

$$\therefore \frac{f}{\omega^2} = c_1 - c_2 \quad \cdots(4)$$

(3)+(4) gives, $\frac{f}{\omega^2} = 2c_1 + \frac{f}{\omega^2}$ $\therefore c_1 = 0$.

From (3), $\therefore c_2 = -\frac{f}{\omega^2}$.

$$\therefore r = -\frac{f}{\omega^2}e^{-\omega t} + \frac{f}{\omega^2}.$$

or, $r = \frac{f}{\omega^2}\left(1-e^{-\omega t}\right) \cdots (5)$

We have $\dot{\theta} = \omega$ or $\frac{d\theta}{dt} = \omega$, $\therefore d\theta = \omega dt$

Integrating, $\theta = \omega t + c$

when $t=0$, $\theta = 0$, $\therefore c = 0$.

$$\therefore \theta = \omega t$$

∴ (5) becomes $r = \frac{f}{\omega^2}(1-e^{-\theta})$ or, $\omega^2 r = f(1-e^{-\theta})$

This is the equation of the path of the ~~particle~~ particle

Radial vel $= \dot r = \frac{f}{\omega}e^{-\omega t}$.

rate of growth of radial vel. $= \frac{d}{dt}(\dot r) = \frac{d}{dt}\left(\frac{f}{\omega}e^{-\omega t}\right) = -f e^{-\omega t}$,

which is not +ve.

When $t \to \infty$, $e^{-\omega t} = \frac{1}{e^{\omega t}} \to 0$.

∴ rate of growth of radial velocity tends to zero (proved)

Ex 7 If the angular vel. about the origin be a const ω, ~~deduce~~ deduce ~~that~~ the cross radial component of the rate of change of accelⁿ of the particle and show that if this rate of change of accelⁿ be zero, then $\frac{d^2r}{dt^2} = \frac{1}{3}\omega^2 r$.

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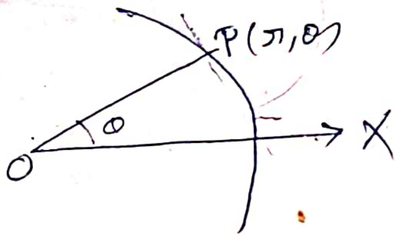
Ans:- Let $P(r,\theta)$ be the position of the particle at time t. Let $\hat\alpha$ and $\hat\beta$ be the unit vectors along OP and ⊥ⁿ to OP.



If f_r and f_θ be the radial and cross radial components of acceleration, ~~Then $f_r = \ddot r - r\omega^2$~~

Then $f_r = \ddot r - r\dot\theta^2 = \ddot r - r\omega^2$

$$f_\theta = \frac{1}{r}\frac{d(r^2\dot\theta)}{dt} = \frac{\omega}{r}\cdot 2r\dot r = 2\omega\dot r$$

Accelⁿ vector $= \vec f = f_r\hat\alpha + f_\theta\hat\beta$

$$\therefore \frac{d}{dt}(\vec f) = \frac{d}{dt}(f_r\hat\alpha + f_\theta\hat\beta) = \frac{d}{dt}(f_r)\hat\alpha + f_r\frac{d\hat\alpha}{dt} + \hat\beta\frac{d}{dt}(f_\theta) + f_\theta\frac{d\hat\beta}{dt}$$

If $\hat i, \hat j$ be the unit vectors along OX and ⊥ⁿ to OX respectively, then

$$\hat\alpha = \hat i\cos\theta + \hat j\sin\theta$$

$$\hat\beta = \hat i\cos(\pi/2+\theta) + \hat j\sin(\pi/2+\theta) = -\hat i\sin\theta + \hat j\cos\theta$$

$$\therefore \frac{d\hat\alpha}{dt} = (-\hat i\sin\theta + \hat j\cos\theta)\dot\theta = \hat\beta\omega$$

$$\frac{d\hat\beta}{dt} = (-\hat i\cos\theta - \hat j\sin\theta)\dot\theta = -\hat\alpha\omega$$

$$\therefore \frac{d\vec f}{dt} = \frac{df_r}{dt}\hat\alpha + f_r\hat\beta\omega + \frac{df_\theta}{dt}\cdot\hat\beta - f_\theta\hat\alpha\omega$$

$$= \left(\frac{df_r}{dt} - f_\theta\omega\right)\hat\alpha + \left(f_r\omega + \frac{df_\theta}{dt}\right)\hat\beta$$

∴ Cross radial component of the rate of change of acceleration

$$= \frac{df_\theta}{dt} + f_r\omega = \frac{d}{dt}(2\omega\dot r) + (\ddot r - r\omega^2)\omega$$

$$= 2\omega\ddot r + \ddot r\omega - r\omega^3 = 3\ddot r\omega - r\omega^3.$$

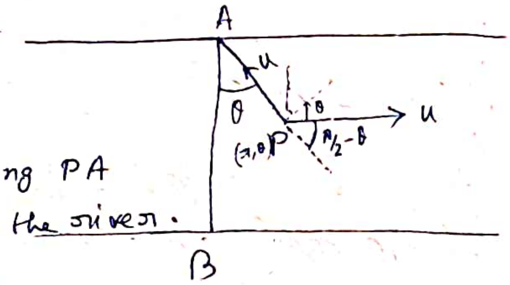
If this component be zero, then,

$$3\ddot r\omega - r\omega^3 = 0$$

or, $\ddot r = \frac{1}{3}r\omega^2$ (proved)

Ex-8 A and B are points on opposite bank of a river of width a and AB is at right angled to the direction of the flow of river. A boat leaves B and is rowed with constant speed u always directed towards A. If the river flows with the speed u, find the path of the boat.

Let $P(r,\theta)$ be the position of the boat at time t, where $AP = r$, $\angle PAB = \theta$.



The boat has two velocities, u along PA and u along the direction of the flow of the river.

Radial velocity is $\frac{dr}{dt} = u\cos(\pi/2-\theta) - u = u\sin\theta - u = u(\sin\theta - 1)$ --- (i)

Cross-radial velocity $= r\frac{d\theta}{dt} = u\sin(\pi/2 - \theta) = u\cos\theta$ --- (ii)

(i) ÷ (ii) gives,

$$\frac{dr}{r\,d\theta} = \frac{\sin\theta - 1}{\cos\theta} = \tan\theta - \sec\theta$$

$$\therefore \frac{dr}{r} = (\tan\theta - \sec\theta)\,d\theta$$

Integrating

$$\therefore \log r = \log\sec\theta - \log(\sec\theta + \tan\theta) + \log c$$

$$\therefore r = \frac{c\sec\theta}{\sec\theta + \tan\theta} = \frac{c}{1+\sin\theta}$$

or $r(1+\sin\theta) = c$.

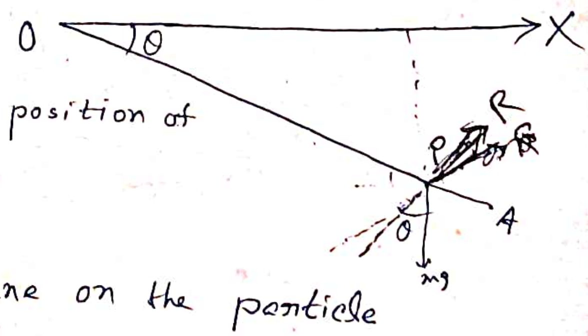
At B, $\theta = 0$, $r = a$, $\therefore c = a$

$\therefore$ The equation of the path is $r(1+\sin\theta) = a$,

Ex-9 A particle is at rest on a smooth horizontal plane, which commences to turn about a st. line lying on itself with constant angular velocity ω downwards. If a be the distance of the particle from the axis of rotation initially, show that, the particle will leave the plane at time t, given by the equation

$$a\sinh\omega t + \frac{g}{2\omega^2}\cosh\omega t = \frac{g}{\omega^2}\cos\omega t.$$

OX is the initial horizontal position of the plane and OA is the position at time t. Then P is the position of the particle. $OP = r$ & $\angle POX = \theta$.



m = mass of the particle

R = Normal pressure of the plane on the particle

The equations of motion are,

$$m\left\{\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right\} = mg\sin\theta \quad \cdots\cdots (1)$$

$$m\cdot\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = -R + mg\cos\theta \quad \cdots (2)$$

By the condition $\frac{d\theta}{dt} = \omega$ or $d\theta = \omega\, dt$ $\therefore \theta = \omega t + c_1$

when $t=0$, $\theta = 0$, $\therefore c_1 = 0$

$\therefore \theta = \omega t$

From (1) $\frac{d^2r}{dt^2} - r\omega^2 = g\sin\omega t$

C.F. is, $c_2\cosh\omega t + c_3\sinh\omega t$

P.I. is $\frac{1}{D^2-\omega^2}\, g\sin\omega t = \frac{g\sin\omega t}{-\omega^2-\omega^2} = -\frac{g\sin\omega t}{2\omega^2}$

The G.S. is

$$r = c_2\cosh\omega t + c_3\sinh\omega t - \frac{g}{2\omega^2}\sin\omega t$$

$$\frac{dr}{dt} = c_2\omega\sinh\omega t + c_3\omega\cosh\omega t - \frac{g}{2\omega^2}\omega\cos\omega t$$

at $t=0$, $r=a$, $\frac{dr}{dt}=0$

$\therefore a = c_2$

And $0 = c_3\omega - \frac{g}{2\omega} \Rightarrow c_3 = \frac{g}{2\omega^2}$

$$\therefore r = a\cosh\omega t + \frac{g}{2\omega^2}(\sinh\omega t - \sin\omega t)$$

From (2), $R = mg\cos\omega t - \frac{m\omega}{r}\, 2r\frac{dr}{dt}$

$$= -2m\omega\left[a\omega\sinh\omega t + \frac{g}{2\omega}(\cosh\omega t - \cos\omega t)\right] + mg\cos\omega t$$

$\therefore$ i.e $R = m\left[2g\cos\omega t - 2a\omega^2\sinh\omega t - g\cosh\omega t\right]$.

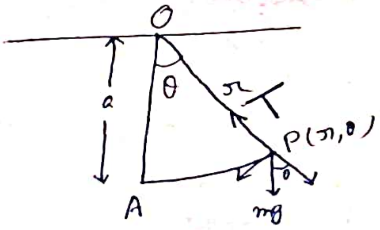
The particle will leave the plane when $R=0$,

ie $2g\cos\omega t - 2a\omega^2\sinh\omega t - g\cosh\omega t = 0$

or, $a\sinh\omega t + \frac{g}{2\omega^2}\cosh\omega t = \frac{g}{\omega^2}\cos\omega t$ (proved)

Ex-10 A heavy particle hangs from a point O by a string of length a. It is projected horizontally with velocity v such that $v^2 = (2+\sqrt{3})ag$. Show that the string becomes slack when it has described an angle $\cos^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

Let P be any position of the particle. The angle described is θ. The particle ~~stars~~ starts from A with a velocity v, which is given by $v^2 = (2+\sqrt{3})ag$.



The equations of motion are given by,

$$m\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right] = mg\cos\theta - T \quad \text{--- (1)}$$

and
$$m\left[\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)\right] = -mg\sin\theta \quad \text{--- (2)}$$

From (2),
$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = -g\sin\theta$$

Since r is a constant and equal to a.

So,
$$\frac{1}{r}\cdot r^2\frac{d^2\theta}{dt^2} = -g\sin\theta \quad \text{or,} \quad r\frac{d^2\theta}{dt^2} = -g\sin\theta \quad \text{or} \quad \frac{d^2\theta}{dt^2} = -\frac{g}{a}\sin\theta$$

Multiplying both sides by $2\frac{d\theta}{dt}$ and integrating we have,

$$\left(\frac{d\theta}{dt}\right)^2 = 2\frac{g}{a}\cos\theta + C \quad \text{--- (3)}$$

Initially,
$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 = (2+\sqrt{3})ag$$

ie
$$\left(\frac{d\theta}{dt}\right)^2 = \frac{(2+\sqrt{3})g}{a} \quad \left[\because r \text{ is a constant} = a\right]$$

So (3) becomes,
$$\frac{(2+\sqrt{3})g}{a} = \frac{2g}{a}\cdot 1 + C \quad [\because \theta = 0]$$

ie
$$C = \frac{2g}{a} - \frac{2g}{a} + \frac{\sqrt{3}g}{a} = \frac{\sqrt{3}g}{a}$$

So (3) becomes,
$$\left(\frac{d\theta}{dt}\right)^2 = \frac{2g}{a}\cos\theta + \frac{\sqrt{3}g}{a} \quad \text{--- (4)}$$

From (1)
$$-mr\left(\frac{d\theta}{dt}\right)^2 = mg\cos\theta - T$$

ie.
$$\left(\frac{d\theta}{dt}\right)^2 = \frac{T}{ma} - \frac{mg\cos\theta}{ma} \quad [\because r = a] \quad \text{--- (5)}$$

From (4) and (5) we have,

$$\frac{2g\cos\theta}{a} + \frac{\sqrt{3}g}{a} = -\frac{g\cos\theta}{a} \quad \left[\because \text{when the string will slack then, } T = 0\right]$$

ie
$$2\cos\theta = -\cos\theta - \sqrt{3}$$

or
$$3\cos\theta = -\sqrt{3} \quad \text{ie} \quad \cos\theta = -\frac{\sqrt{3}}{3} = -\frac{1}{\sqrt{3}}$$

ie
$$\theta = \cos^{-1}\left(-\frac{1}{\sqrt{3}}\right)$$